

A SYMMETRICAL NOTATION FOR NUMBERS

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The possibility of representing real numbers in various scales of notation is well known. Thus, in the scale r an arbitrary positive number b may be expanded in the form,

$$b = \sum_{-\infty}^N a_n r^n, \quad 0 \leq a_n \leq r - 1,$$

and represented in the "decimal" notation as $a_N a_{N-1} \cdots a_0 \cdot a_{-1} a_{-2} \cdots$. Negative numbers are represented by prefixing a minus sign to the representation of the corresponding positive numbers. Although it seems unlikely that the scale ten will ever be changed for ordinary work, the use of other scales and systems of notation is still of practical as well as mathematical interest. In some types of computing machines, for example, scales other than ten lend themselves more readily to mechanization.

A slight modification of the ordinary expansion gives a representation for numbers with certain computational advantages. Assuming r to be odd, it is seen easily that any positive or negative number b can be represented as

$$b = \sum_{-\infty}^N a_n r^n, \quad -\frac{r-1}{2} \leq a_n \leq \frac{r-1}{2},$$

and we may denote b as usual by the sequence of its digits

$$b = a_N \cdots a_0 \cdot a_{-1} \cdots$$

Both positive and negative numbers are thus represented by a standard notation without a prefixed sign, the sign being implied by the digits themselves; the number is positive or negative according as the first (nonvanishing) digit is greater or less than zero. Every real number has a unique representation apart from those whose expansion ends in an infinite sequence of the digits $(r-1)/2$ or $-(r-1)/2$, each of which has two representations. If this notation were to be used, a simple notation should be invented for the negative digits which suggested their close relation to the corresponding positive digits. For typographical simplicity we shall here denote the negative digits by placing primes on the corresponding positive digits. The notation for the first nine positive and negative integers with $r=3, 5, 7, 9$ is as follows:

r	-9	-8	-7	-6	-5	-4	-3	-2	-1
3	1'00	1'01	1'11'	1'10	1'11	1'1'	1'0	1'1	1'
5	2'1	2'2	1'2'	1'1'	1'0	1'1	1'2	2'	1'
7	1'2'	1'1'	1'0	1'1	1'2	1'3	3'	2'	1'
9	1'0	1'1	1'2	1'3	1'4	4'	3'	2'	1'

r	0	1	2	3	4	5	6	7	8	9
3	0	1	11'	10	11	11'1'	11'0	11'1	101'	100
5	0	1	2	12'	11'	10	11	12	22'	21'
7	0	1	2	3	13'	12'	11'	10	11	12
9	0	1	2	3	4	14'	13'	12'	11'	10

In general the negative of any number is found by placing a prime on each unprimed digit and taking it off each primed digit. Arithmetic operations with this system are considerably simplified. In the first place the symmetries introduced by this notation make the addition and multiplication tables much easier to learn. For the scale $r=9$ these tables are, respectively, as follows:

+	4'	3'	2'	1'	0	1	2	3	4
4'	1'1	1'2	1'3	1'4	4'	3'	2'	1'	0
3'	1'2	1'3	1'4	4'	3'	2'	1'	0	1
2'	1'3	1'4	4'	3'	2'	1'	0	1	2
1'	1'4	4'	3'	2'	1'	0	1	2	3
0	4'	3'	2'	1'	0	1	2	3	4
1	3'	2'	1'	0	1	2	3	4	14'
2	2'	1'	0	1	2	3	4	14'	13'
3	1'	0	1	2	3	4	14'	13'	12'
4	0	1	2	3	4	14'	13'	12'	11'

.	4'	3'	2'	1'	0	1	2	3	4
4'	22'	13	11'	4	0	4'	1'1	1'3'	2'2
3'	13	10	13'	3	0	3'	1'3	1'0	1'3'
2'	11'	13'	4	2	0	2'	4'	1'3	1'1
1'	4	3	2	1	0	1'	2'	3'	4'
0	0	0	0	0	0	0	0	0	0
1	4'	3'	2'	1'	0	1	2	3	4
2	1'1	1'3	4'	2'	0	2	4	13'	11'
3	1'3'	1'0	1'3	3'	0	3	13'	10	13
4	2'2	1'3'	1'1	4'	0	4	11'	13	22'

The labor in learning the tables would appear to be reduced by a factor of at least two from the corresponding $r=9$ case in ordinary notation. There is no need to learn a "subtraction table"; to subtract, one primes all digits of the subtrahend and adds. The sign of the difference automatically comes out correct, and the clumsy device of "borrowing" is unnecessary. More generally, to add a set of numbers, some positive and some negative, all are placed in a column without regard to sign and added, *e.g.* ($r=9$):

(1')		(1)		carried numbers.
1	3'	1'	2	
2'	3	1	4	
4'	1'	2	3	
3	2'	3'	4	
3	0	0	1'	
1'	2'	1	3'	
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1'	4	1	0	

This process may be contrasted with the usual method where the positive and negative numbers must be added separately, the smaller sum subtracted from the larger and the difference given the sign of the larger, that is, three addition or subtraction processes and a sign rule, while with the symmetrical system one standard addition process covers all cases. Furthermore, in such a sum cancellation is very common and reduces considerably the size of numbers to be carried in memory in adding a column; this follows from the fact that any digit cancels its negative and these may be struck out from a column without affecting the sum. If all digits are equally likely and independent, the sum in a column will have a mean value zero, standard deviation $\sqrt{p(r^2-1)/12}$ where p is the number of numbers being added, while in the usual notation the mean value is $p(r/2)$ with the same standard deviation.

Multiplication and division may be carried out also by the usual processes, and here again signs take care of themselves, although in these cases, of course, the advantage of this is not so great.

We may note also that in the usual system of notation, when we wish to "round off" a number by replacing all digits after a certain point by zeros, the digits after this point must be inspected to see whether they are greater or less than 5 in the first place following the point. In the former case the preceding digit is increased by one. With the symmetrical system one always obtains the closest approximation merely by replacing the following digits by zeros. Numbers such as $1.444 \dots = 2.4'4'4' \dots$ with two representations are exactly half way between the two nearest rounded off approximations, and in this case we obtain the upper or lower approximation depending on which representation is rounded off. If we were using this notation, department stores would find it much more difficult to camouflage the price of goods with \$.98 labels.

We have assumed until now that the scale r is odd. If r is even, say 10, a slightly unbalanced system of digits can be used; for example, $4', 3', 2', 1', 0, 1, 2, 3, 4, 5$. The dissymmetry introduced unfortunately loses several of the advantages described above, *e.g.*, the ease of negation and hence of subtraction, and also the round off property.

A more interesting possibility is that of retaining symmetry by choosing for "digits" numbers halfway between the integers. In the case $r=10$ the possible digits would be

$$a_n = \frac{9'}{2}, \frac{7'}{2}, \frac{5'}{2}, \frac{3'}{2}, \frac{1'}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2},$$

and any number b can be expressed as

$$b = \sum_{-\infty}^N a_n r^n.$$

In this system the properties of positive-negative symmetry, automatic handling of signs, and simple round off are retained. One curious and disadvantageous feature is that the integers can only be represented as infinite decimals, and this is possible in an infinite number of different ways. For example.

$$0 = \frac{1}{2} \frac{9'}{2} \frac{9'}{2} \dots = \frac{1}{2} \frac{9'}{2} \frac{9'}{2} \frac{9'}{2} \dots = \frac{1'}{2} \frac{9}{2} \frac{9}{2} \frac{9}{2} \dots \text{etc.}$$

Symmetrical notation offers attractive possibilities for general purpose computing machines of the electronic or relay types. In these machines it is possible to perform the calculations in any desired scale and only translate to the scale ten at input and output. The use of a symmetrical notation simplifies many of the circuits required to take care of signs in addition and subtraction, and to properly round off numbers.
